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Auxetic behaviour from stretching connected squares

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Abstract Systems with negative Poisson's ratio (auxetic) exhibit the unusual characteristic of getting fatter when stretched and thinner when compressed. Such behaviour is a scale-independent property and is the result of a cooperation between the internal geometry of the system and the way this deforms when uniaxially stretched. Here, we analyse the anisotropic mechanical properties for a system constructed from connected squares which can deform through changes in length of the sides of the squares (idealised 'stretching squares' model). In particular, we show that this system may exhibit a negative Poisson's ratio which depends on the angle between the squares and the direction of loading but is independent of the size of the squares which suggests that this model may be implemented at any scale of structure including the micro- and nano-level. We also show how this model compares and complements the existing 'rotating squares' model which also works on a system with the same geometric characteristics and which has been shown to lead to auxeticity in various classes of materials.

Introduction

Auxetics exhibit the unusual property of expanding in width when stretched and becoming thinner when compressed, i.e. they exhibit a negative Poisson's ratio [1]. Although this property is not exhibited in most everyday materials (commonly used materials have a positive Poisson's ratio), in recent years there has been considerable work in the field of auxetics which led to the discovery of this unusual property in various types of materials and structures [1–34], some of which (e.g. the proposed re-entrant molecular-level honeycombs [1]) still need to be synthesised. In all of the auxetics discovered so far, it has been found that this unusual property can be explained in terms of geometric features of the structure (the micro/nanostructure in the case of materials) and the way this structure deforms when subjected to uniaxial loads (the deformation mechanism). In fact, research on auxetics has been characterised by the development of various idealised models that could be used to explain the presence of auxeticity. Such models include the re-entrant honeycomb model which was successfully used to model the behaviour of auxetic microstructured foams [9–14], 'nodes and fibrils' models which have been used to describe the behaviour of auxetic microstructured polymers [15–17], models based on rigid 'free' molecules [6–8], models of various chiral structures [4, 5] and models of systems made from 'rotating rigid units' such as squares [35–37], triangles [38], rectangles [14, 39, 40] or tetrahdera [27] which have proved to be particularly suitable for explaining the auxetic behaviour in various classes of materials ranging from foams [9–14] to silicates [24–32] and zeolites [33, 34].

A 'rotating rigid units' model system, which in recent years has attracted considerable attention, is the one made from 'connected squares', which when stretched, the squares rotate relative to each other as illustrated in Fig. 1a [35–37]. Analytical modelling of this idealised 'rotating connected squares' system, where the squares are assumed to be perfectly rigid and simply rotate relative to each other, suggests that this system has in-plane on-axis

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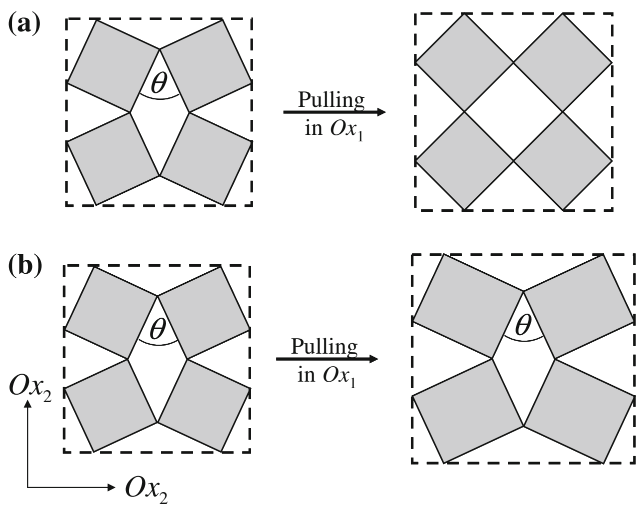

Fig. 1 (**a**) The 'rotating squares' mechanism where the squares rotate relative to each other, i.e. there is a change in the angle θ, and (**b**) the 'stretching squares' mechanism, i.e. the angle θ remains constant but the lengths of the sides of the squares change

Poisson's ratios, Young's moduli and compliance matrix given by:

$$\nu_{21}^{\mathrm{R}} = \left(\nu_{12}^{\mathrm{R}}\right)^{-1} = -1 \tag{1}$$

$$E_1^{\mathrm{R}} = E_2^{R} = \frac{8K_h}{l^2 z[1-\sin(\theta)]} \tag{2}$$

$$\mathbf{S}^{\mathrm{R}} = \frac{l^2 z[1-\sin(\theta)]}{8K_h}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3}$$

where the superscript 'R' is used to indicate that these expressions refer to the rotation mechanism, l is the length of a side of the square, z is the out of plane thickness of the structure, K_h the stiffness coefficient for the hinges and θ the angle between the squares. Note that this system has an infinite shear modulus due to the assumed perfect rigidity of the square elements.

These expressions indicate that the Poisson's ratio of this system is independent of the initial geometrical conformation, i.e. the angles between the squares and the size of the squares. Furthermore, an off-axis analysis of the Poisson's ratio of this system suggests that the in-plane Poisson's ratio, when defined,[1] is always equal to –1, i.e. it is independent of the direction of loading. The expressions also suggest that the Young's modulus has a finite value unless $\theta = \frac{\pi}{2}$, in which case the Young's modulus diverges to infinity. This corresponds to the fully opened structure, and signifies that it cannot be opened any further (see footnote 1).

[1] The Poisson's ratio of the idealised rotating squares system is undefined when the squares are at 90° to each other, i.e. when the structure is fully open and cannot expand any further. At this point, the Young's modulus of the idealised hinging model is infinite.

Despite the simplicity of the rotating squares model, it has been shown that this mechanism plays a very important role for generating auxetic behaviour at various scales, including the nano- (molecular) level. In fact, force-field based molecular modelling simulations have shown that this deformation mechanism can qualitatively explain the negative Poisson's ratio in various zeolites such as natrolite (NAT) and thomsonite (THO) [41, 42]. Nevertheless, as discussed elsewhere [41, 42], in real systems such as NAT and THO, the idealised rotating squares model is too simplistic to capture the complex deformations that occur when these systems are uniaxially stretched, and in fact, a more complex model has been derived where the squares are treated as semi-rigid objects which change shape whilst rotating relative to each other [41, 42].

In this work we investigate through analytical modelling the properties of systems made from squares connected together from their corners (as illustrated in Fig. 1) which deform solely through changes in length of the sides of the squares (henceforth referred to as the 'stretching squares' model, see Figs. 1b and 2). In other words, we discuss the behaviour of a model based on squares which are connected with an identical topology as the 'rotating squares' system (see Fig. 1a). In this model, the sides of the squares are constructed from piston-like elements at a fixed angle to each other which permit the squares to change their side lengths to become rectangles (or squares of different sizes) without changing the angles in the system.

Analytical model

In this section we derive equations to describe the in-plane mechanical properties of the stretching squares system illustrated in Fig. 1b.

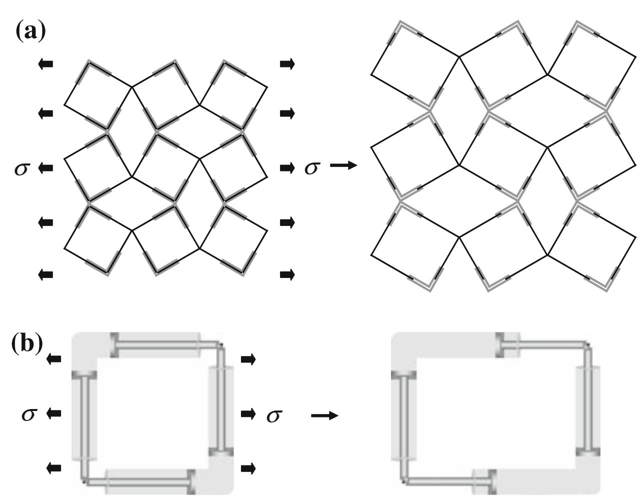

Fig. 2 (**a**) An illustration of the 'stretching squares' mechanism for a structure constructed from 'squares' having sides made from piston-like elements as shown in (**b**)

As shown in Fig. 3 this 'connected squares' system may be described by two unit cells one of which (Fig. 3a) contains four squares and the other (Fig. 3b) two squares per unit cell. It is easy to show that the system in Fig. 3a (henceforth referred to as Orientation I) is oriented at 45° to the system in Fig. 3b (henceforth referred to as Orientation II) and that in these orientations, the properties for loading in the Ox_1 direction are the same as the properties for loading in the Ox_2 direction due to the symmetry of the structures.

For both Orientations I and II (and only these orientations), uniaxial loading of this system in any of the Ox_1 and Ox_2 directions will not result in a shear strain and thus, the elements s_{13} and s_{23} of the compliance matrices (and from symmetry requirements of the matrices, s_{31} and s_{32}) are zero, i.e.:

$$s_{13}^{\mathrm{I}} = s_{23}^{\mathrm{I}} = s_{31}^{\mathrm{I}} = s_{32}^{\mathrm{I}} = 0 \quad (4)$$

$$s_{13}^{\mathrm{II}} = s_{23}^{\mathrm{II}} = s_{31}^{\mathrm{II}} = s_{32}^{\mathrm{II}} = 0 \quad (5)$$

where in order to distinguish between the properties of the system in the two orientations, we have used the superscripts I and II to refer to the properties of the system in Orientations I and II, respectively. (This nomenclature will be used throughout this paper.)

The existence of these two unit cells (one oriented at 45° to the other) and the symmetry of the system provide a simplified method for deriving the full (3 × 3) compliance matrix of these systems. In fact, as shown in Appendix A, we may derive the Young's moduli and Poisson's ratios for one orientation and, with the help of standard transformation techniques, use these to obtain the shear properties of the other orientation [43, 44] *via* the following relations[2]:

$$G_{12}^{\mathrm{II}} = \frac{E_1^{\mathrm{I}}}{2\left(1+\nu_{12}^{\mathrm{I}}\right)} \;\&\; G_{12}^{\mathrm{I}} = \frac{E_1^{\mathrm{II}}}{2\left(1+\nu_{12}^{\mathrm{II}}\right)} \quad (6)$$

where G_{12} is the on-axis shear modulus, E_1 is the Young's modulus for loading in the Ox_1 direction and ν_{12} is the Poisson's ratio in the Ox_1–Ox_2 plane for loading in the Ox_1 direction. As stated above, the superscripts I and II are used to refer to the properties of the system in Orientations I and II, respectively.

In view of all this, to derive the full (3 × 3) compliance matrix for these systems we will first obtain the on-axis Poisson's ratios and Young's moduli for Orientations I and II and then use these to infer the shear moduli. In this way we will obtain all the non-zero elements of the compliance matrix for the properties in the two orientations.

[2] It is important to note that as discussed in Appendix A, these relations are only applicable to systems where (1) the two orientations for which we know the properties are at 45° to each other, (2) for both orientations, the structure should have zero shear coupling coefficients and (3) the systems must exhibit '2D cubic symmetry', i.e. $s_{11} = s_{22}$.

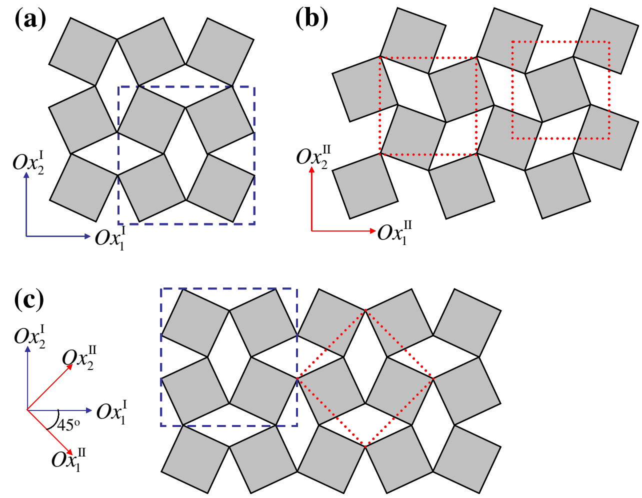

Fig. 3 The unit cells used to derive the mechanical properties of the *'stretching squares'* model for (**a**) Orientation I and (**b**) Orientation II. (**c**) The geometric relation between the two orientations

The on-axis Poisson's ratios and Young's moduli for Orientation I

To derive the in-plane on-axis mechanical properties for loading in the Ox_1 direction for Orientation I, from symmetry, we may simply consider the properties of one-fourth of the unit cell given in Fig. 1a, which is specifically illustrated in Fig. 4. For this system the dimensions of the projections of one-fourth of the larger unit cell with Orientation I in the Ox_i^{I} directions, the X_i^{I}'s, are given by:

$$X_1^{\mathrm{I}} = X_2^{\mathrm{I}} = X^{\mathrm{I}} = l\left[\sin\left(\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right)\right] = l[\sin(\phi) + \cos(\phi)] \quad (7)$$

$$X_3^{\mathrm{I}} = z \quad (8)$$

where l is the length of the sides of the square; θ is the angle between the squares; and ϕ is the angle between the squares and the Ox_i^{I} axis (i = 1, 2) that, for non-sheared unit cells, corresponds to half θ as indicated in Fig. 4.

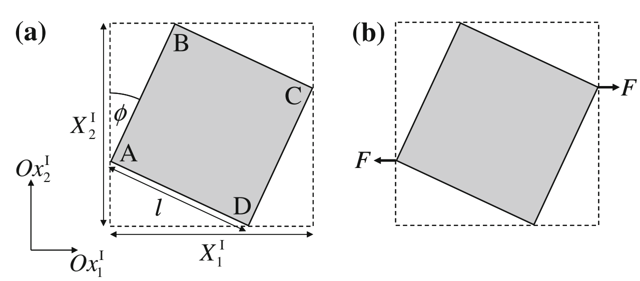

Fig. 4 One-fourth of an Orientation I unit cell showing (**a**) dimensions and (**b**) forces

If we consider a unit cell being subjected to a load σ_1^{I} in the Ox_1^{I} direction, referring to Fig. 4, the forces acting in the direction of the beams AB, CD, BC and DA are given by:

$$P_{\mathrm{AB}}^{\mathrm{I}} = P_{\mathrm{CD}}^{\mathrm{I}} = F^{\mathrm{I}} \sin(\phi) \quad (9)$$

and

$$P_{\mathrm{BC}}^{\mathrm{I}} = P_{\mathrm{AD}}^{\mathrm{I}} = F^{\mathrm{I}} \cos(\phi) \quad (10)$$

where the subscripts indicate along which beam the force is acting and F^{I} relates to σ_1^{I} through:

$$F^{\mathrm{I}} = \frac{1}{2}\sigma_1^{\mathrm{I}} X_2^{\mathrm{I}} X_3^{\mathrm{I}} \quad (11)$$

Therefore, the changes in the length of the beams AB, CD, BC and DA due to the stress σ_1^{I} are given by:

$$\delta_{\mathrm{AB}}^{\mathrm{I}} = \delta_{\mathrm{CD}}^{\mathrm{I}} = \frac{P_{\mathrm{AB}}^{\mathrm{I}}}{K_{\mathrm{s}}} = \frac{P_{\mathrm{CD}}^{\mathrm{I}}}{K_{\mathrm{s}}} = \frac{\sigma_1^{\mathrm{I}} X_2^{\mathrm{I}} X_3^{\mathrm{I}} \sin(\phi)}{2K_{\mathrm{s}}} \quad (12)$$

and

$$\delta_{\mathrm{BC}}^{\mathrm{I}} = \delta_{\mathrm{DA}}^{\mathrm{I}} = \frac{P_{\mathrm{BC}}^{\mathrm{I}}}{K_{\mathrm{s}}} = \frac{P_{\mathrm{DA}}^{\mathrm{I}}}{K_{\mathrm{s}}} = \frac{\sigma_1^{\mathrm{I}} X_2^{\mathrm{I}} X_3^{\mathrm{I}} \cos(\phi)}{2K_{\mathrm{s}}} \quad (13)$$

where K_{s} is the stretching force constant per unit length of the beams.

The strains in the Ox_1^{I} and Ox_2^{I} directions may be defined in terms of the displacements of the individual beams AB and BC (or CD and DA) as follows:

$$\varepsilon_1^{\mathrm{I}} = \frac{\Delta X_1^{\mathrm{I}}}{X_1^{\mathrm{I}}} = \frac{\delta_{\mathrm{AB}}^{\mathrm{I}} \sin(\phi) + \delta_{\mathrm{BC}}^{\mathrm{I}} \cos(\phi)}{X_1^{\mathrm{I}}} \quad (14)$$

and

$$\varepsilon_2^{\mathrm{I}} = \frac{\Delta X_2^{\mathrm{I}}}{X_2^{\mathrm{I}}} = \frac{\delta_{\mathrm{AB}}^{\mathrm{I}} \cos(\phi) + \delta_{\mathrm{BC}}^{\mathrm{I}} \sin(\phi)}{X_2^{\mathrm{I}}} \quad (15)$$

where since $X_1^{\mathrm{I}} = X_2^{\mathrm{I}} = X^{\mathrm{I}}$ and $X_3^{\mathrm{I}} = z$ these strains simplify to:

$$\varepsilon_1^{\mathrm{I}} = \frac{\Delta X_1^{\mathrm{I}}}{X_1^{\mathrm{I}}} = \frac{z}{2K_{\mathrm{s}}}\sigma_1^{\mathrm{I}} \quad (16)$$

and

$$\varepsilon_2^{\mathrm{I}} = \frac{\Delta X_2^{\mathrm{I}}}{X_2^{\mathrm{I}}} = \frac{z\cos(\phi)\sin(\phi)}{K_{\mathrm{s}}}\sigma_1^{\mathrm{I}} = \frac{z\sin(2\phi)}{2K_{\mathrm{s}}}\sigma_1^{\mathrm{I}} = \frac{z\sin(\theta)}{2K_{\mathrm{s}}}\sigma_1^{\mathrm{I}} \quad (17)$$

Thus, the Poisson's ratio and Young's modulus for loading in the Ox_1^{I} direction are given by:

$$\nu_{12}^{\mathrm{I}} = -\frac{\varepsilon_2^{\mathrm{I}}}{\varepsilon_1^{\mathrm{I}}} = -\sin(\theta) \quad (18)$$

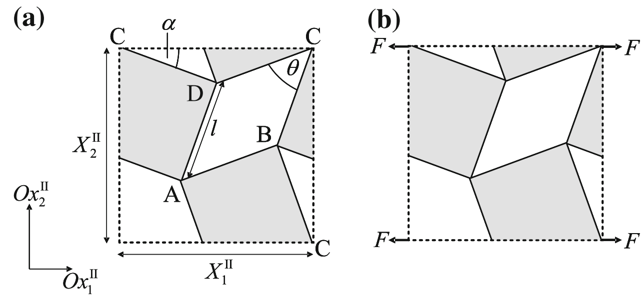

Fig. 5 An Orientation II unit cell showing (**a**) dimensions and (**b**) forces

$$E_1^{\mathrm{I}} = \frac{\sigma_1^{\mathrm{I}}}{\varepsilon_1^{\mathrm{I}}} = \frac{2K_{\mathrm{s}}}{z} \quad (19)$$

Note that, from symmetry, the Poisson's ratio and Young's modulus for loading in the Ox_2^{I} direction are equal to their respective values for loading in the Ox_1^{I} direction, i.e.:

$$\nu_{21}^{\mathrm{I}} = \nu_{12}^{\mathrm{I}} = -\sin(\theta) \quad (20)$$

$$E_2^{\mathrm{I}} = E_1^{\mathrm{I}} = \frac{2K_{\mathrm{s}}}{z} \quad (21)$$

The on-axis Poisson's ratios and Young's moduli for Orientation II

Here we consider the unit cell illustrated in Fig. 5. For this system, the projections of the undeformed squares in the Ox_i^{II} directions, the X_i^{II}'s, are given by:

$$X_1^{\mathrm{II}} = X_2^{\mathrm{II}} = X^{\mathrm{II}} = 2l\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = 2l\cos(\alpha) \quad (22)$$

$$X_3^{\mathrm{II}} = z \quad (23)$$

where α is the acute angle between the sides of the squares and the Ox_i^{II} directions ($i = 1, 2$) as shown in Fig. 5a.

The force acting on the sides of the squares and their corresponding change in length can be obtained in a similar manner as that used for Orientation I. However, this time the whole unit cell needs to be considered. As illustrated in Fig. 5b, only the beams parallel to CD will experience a change in length as a result of a stress σ_1^{II} in the Ox_1^{II} direction, i.e. the changes in lengths in the beams are given by:

$$\delta_{\mathrm{BC}}^{\mathrm{II}} = \delta_{\mathrm{AD}}^{\mathrm{II}} = 0 \quad (24)$$

and

$$\delta_{\mathrm{CD}}^{\mathrm{II}} = \delta_{\mathrm{AB}}^{\mathrm{II}} = \frac{P_{\mathrm{CD}}^{\mathrm{II}}}{K_{\mathrm{s}}} = \frac{P_{\mathrm{AB}}^{\mathrm{II}}}{K_{\mathrm{s}}} = \frac{1}{2K_{\mathrm{s}}}\sigma_1^{\mathrm{II}} X_2^{\mathrm{II}} X_3^{\mathrm{II}} \cos(\alpha) \quad (25)$$

where

$$P^{II}_{CD} = P^{II}_{AB} = F^{II}\cos(\alpha)$$

where F^{II} relates to σ^{II}_1 through:

$$F^{II} = \frac{1}{2}\sigma^{II}_1 X^{II}_2 X^{II}_3 \qquad (26)$$

The strains in the Ox^{II}_1 and Ox^{II}_2 directions may be defined in terms of the change in lengths of the single beams as before:

$$\varepsilon^{II}_1 = \frac{\Delta X^{II}_1}{X^{II}_1} = \frac{2\delta^{II}_{CD}\cos(\alpha)}{X^{II}_1} = \frac{z\cos^2(\alpha)}{K_s}\sigma^{II}_1 \qquad (27)$$

$$\varepsilon^{II}_2 = \frac{\Delta X^{II}_2}{X^{II}_2} = \frac{2\delta^{II}_{BC}\sin(\alpha)}{X^{II}_2} = 0 \qquad (28)$$

since $X^{II}_1 = X^{II}_2 = X^{II}$ and $X^{II}_3 = z$.

Thus, the Poisson's ratio and Young's modulus for loading in the Ox^{II}_1direction are given by:

$$\nu^{II}_{12} = -\frac{\varepsilon^{II}_2}{\varepsilon^{II}_1} = 0 \qquad (29)$$

and

$$E^{II}_1 = \frac{\sigma^{II}_1}{\varepsilon^{II}_1} = \frac{K_s}{z\cos^2(\alpha)} = \frac{K_s}{z\left[\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\theta}{2}\right)\right]^2} = \frac{2K_s}{z[1+\sin(\theta)]} \qquad (30)$$

Note that, once again, from symmetry, the Poisson's ratio and Young's modulus for loading in the Ox^{II}_2 direction are equal to their respective values for loading in the Ox^{II}_1 direction.

The on-axis shear behaviour

As discussed above and shown in Appendix A, the shear modulus for Orientation I can be related to the on-axis properties for loading in Orientation II to obtain:

$$G^{I}_{12} = \frac{E^{II}_1}{2(1+\nu^{II}_{12})} = \frac{K_s}{2z\cos^2(\alpha)} = \frac{K_s}{z[1+\sin(\theta)]} \qquad (31)$$

and similarly, the shear modulus for Orientation II can be related to the on-axis properties for loading in Orientation I to obtain:

$$G^{II}_{12} = \frac{E^{I}_1}{2\left(1+\nu^{I}_{12}\right)} = \frac{K_s}{z[1-\sin(\theta)]} \qquad (32)$$

Summary and off-axis properties

To summarise, the on-axis mechanical properties of the two orientations are given by the (3 × 3) compliance matrices which relates stress to strain through $\boldsymbol{\varepsilon} = \mathbf{S}\boldsymbol{\sigma}$ where for Orientation I, $\mathbf{S}$ is given by:

$$\mathbf{S}^{I} = \frac{z}{2K_s}\begin{pmatrix} 1 & \sin(\theta) & 0 \\ \sin(\theta) & 1 & 0 \\ 0 & 0 & 2+2\sin(\theta) \end{pmatrix} \qquad (33)$$

and for Orientation II, $\mathbf{S}$ is given by:

$$\mathbf{S}^{II} = \frac{z}{2K_s}\begin{pmatrix} 1+\sin(\theta) & 0 & 0 \\ 0 & 1+\sin(\theta) & 0 \\ 0 & 0 & 2-2\sin(\theta) \end{pmatrix} \qquad (34)$$

Using standard transformation techniques [44] and taking the on-axis mechanical properties to be in Orientation I, it is easy to show that for the stretching mechanism the Poisson's ratio ν^{ζ}_{12}, Young's moduli E^{ζ}_1 and shear modulus G^{ζ}_{12} at an angle ζ to the Ox_1-axis are given by:

$$\nu^{\zeta}_{12} = -\frac{\cos^2(2\zeta)\sin(\theta)}{1+\sin^2(2\zeta)\sin(\theta)} \qquad (35)$$

$$E^{\zeta}_1 = \frac{2K_s}{z\left[1+\sin^2(2\zeta)\sin(\theta)\right]} \qquad (36)$$

$$G^{\zeta}_{12} = \frac{K_s}{z[1+\cos(4\zeta)\sin(\theta)]} \qquad (37)$$

where the superscript ζ is used to indicate that these quantities refer to the off-axis property. Note that if ζ is set equal to 45°, these expressions reduce to the respective expressions for Orientation II as expected.

Results and discussion

The on-axis Poisson's ratios, Young' moduli and shear moduli of this system for various values of the angle θ between the squares when the system is in Orientations I and II are shown in Fig. 6, whilst plots of their respective off-axis quantities are found in Figs. 7–9.

These plots (Figs. 6–9) and expressions derived above for the mechanical properties at infinitesimally small strains clearly suggest that although the mechanical parameters obtained from the stretching squares mechanism are remarkably different from those of the corresponding rotating squares mechanism, this new 'stretching squares' system is still capable of exhibiting a negative Poisson's ratio. In this respect we note that this behaviour is in sharp contrast with that of re-entrant and non-re-entrant honeycombs which are well known for their auxetic properties [45] where it has been shown that the hinging mode of deformation always produces Poisson's ratios which are of opposite signs to that of the stretching mode of deformation.

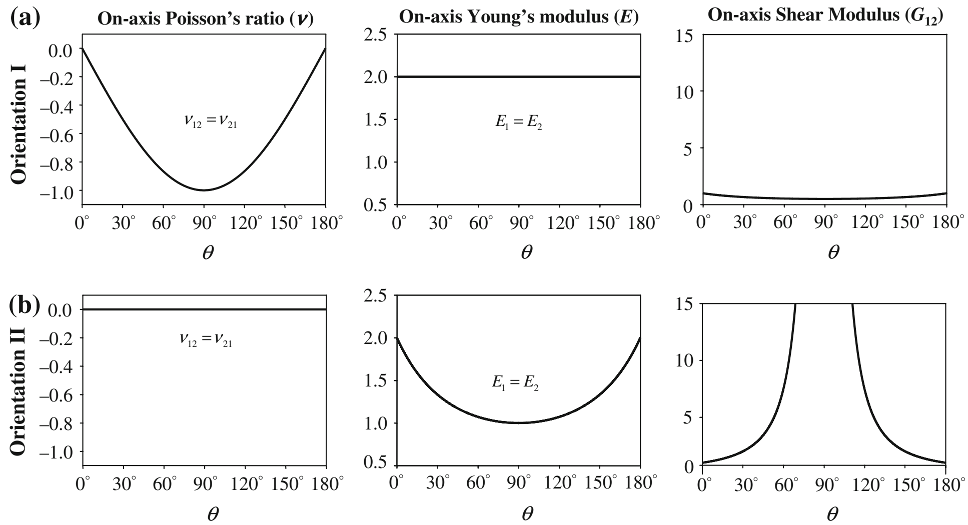

Fig. 6 The variation of the Poisson's ratios, Young's moduli and shear moduli with the angle between the squares θ when the system is loaded in (**a**) Orientation I and (**b**) Orientation II. It is assumed that $K_s = 1$ and $z = 1$

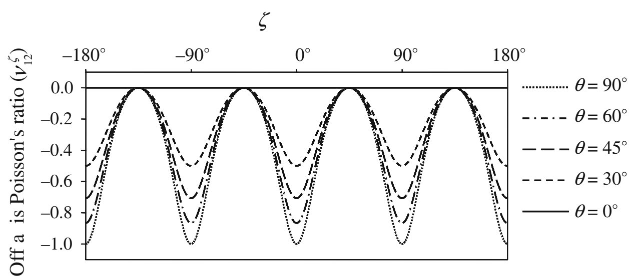

Fig. 7 Off-axis plots for the Poisson's ratio of the stretching mechanism at varying degrees of openness. $\zeta = 0^o$ corresponds to Orientation I. It is assumed that $K_s = 1$ and $z = 1$

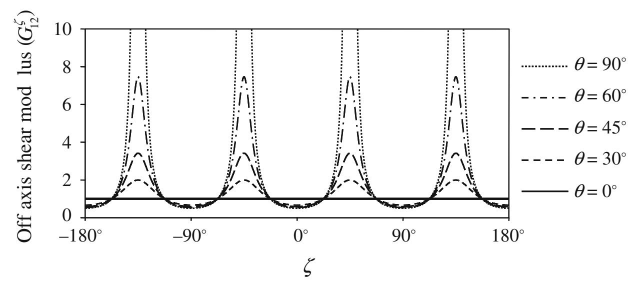

Fig. 9 Off-axis plots for the shear modulus of the stretching mechanism at varying degrees of openness. $\zeta = 0^o$ corresponds to Orientation I. It is assumed that $K_s = 1$ and $z = 1$

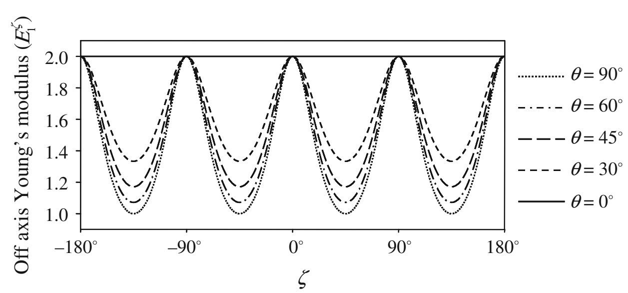

Fig. 8 Off-axis plots for the Young's modulus of the stretching mechanism at varying degrees of openness. $\zeta = 0^o$ corresponds to Orientation I. It is assumed that $K_s = 1$ and $z = 1$

Considering first the Poisson's ratios for this novel 'stretching squares' model, we note that although the on-axis and off-axis Poisson's ratios are independent of the size of the squares, they are dependent on:

(i) the angle between the squares $\theta \in [0, 180^o]$, or equivalently, the degree of openness of the structure;
(ii) the direction of loading $\zeta \in [0, 360^o]$

and can assume values between 0 and –1 (both values included). This suggests that a square network that deforms through a stretching mechanism can be adjusted to assume specific pre-determined values of the Poisson's ratio by adjusting both the openness (θ) of the structure and/or the orientation of the structure relative to the direction of the applied stress. This is not possible with the rotating squares mechanism due to the isotropic nature of its Poisson's ratio (although isotropic systems are also extremely useful in their own accord) irrespective of the degree of openness of the structure (i.e. the angle between the squares). In this respect, it is important to note that although the stretching mechanism removes the isotropy that is shown by the rotation mechanism, the stretching squares system is still characterised by a high degree of symmetry since the structure is highly symmetric (it belongs to the P4 g group) which confers a rotational symmetry of order 4 to the mechanism. This can be easily seen from the shape of the compliance matrices which have $s_{11} = s_{22}$ and can be easily inferred from the expressions of the off-axis properties.[3]

[3] Equations 35 and 36 have the property of being periodic with respect to ζ with a period equal to 90° giving $\nu_{12}^{\zeta}(\zeta) = \nu_{12}^{\zeta}(\zeta + 90^o) = \nu_{21}^{\zeta}$ and $E_1^{\zeta}(\zeta) = E_1^{\zeta}(\zeta + 90^o) = E_2^{\zeta}(\zeta)$.

Also, the range of values that the Poisson's ratio can attain is finite and an upper and lower bound for this idealised system can be inferred from the off-axis expression of this quantity (Eq. 35, plotted in Fig. 7). This expression suggests that for $\theta \in [0, 180^{o}]$ and $\zeta \in [0, 360^{o}]$, the term $-\cos^2(2\zeta)\sin(\theta)$ (found in the numerator of Eq. 35) takes values between -1 and 0, whilst the term $1 + \sin^2(2\zeta)\sin(\theta)$ (a term found in the denominator of Eq. 35) takes values in between 1 and 2. Thus, since the sign of the numerator is always negative whilst the sign of the denominator is always positive and the numerator has a magnitude that is always smaller than or equal to that of the denominator, the Poisson's ratio for any conformation and direction of loading will have a value between 0 and -1. These boundaries are clearly illustrated in the off-axis plots in Fig. 7.

The locations of the maxima and minima can be derived from Eq. 35 by first differentiating ν_{12}^{ζ} with respect to ζ, setting the result equal to zero in order to determine the location of the turning points and then looking at the second derivatives to establish the nature of these turning points (see Appendix B). This reveals:

(i) a minimum when $\zeta = 0, \pm 90^{o}, \pm 180^{o}$, i.e. when the structure is loaded on-axis in Orientation I, which corresponds to Poisson's ratio of $-\sin(\theta)$, and
(ii) a maximum of zero when $\zeta = \pm 45^{o}, \pm 135^{o}$, i.e. when the structure is loaded on-axis in Orientation II.

Thus, the most auxetic configuration exhibited by this system occurs when $\theta = 90$ with $\zeta = 0, \pm 90^{o}, \pm 180^{o}$ (on-axis in Orientation I) giving the upper bound Poisson's ratio of -1.

The points when $\zeta = \pm n 45^{o}\, n \in \mathbb{Z}$ also correspond to extreme values of the Young's moduli and shear moduli where at $\zeta = 0, \pm 90^{o}, \pm 180^{o}$ (on-axis in Orientation I), the moduli are at a minimum whilst when $\zeta = \pm 45^{o}, \pm 135^{o}$ (on-axis in Orientation II) the moduli are at a maximum. It is important to note that, with the exception of the system with $\theta = 90^{o}$, where the shear moduli are infinite when $\zeta = \pm 45^{o}, \pm 135^{o}$ (on-axis in Orientation II),[4] the Young's and shear moduli of the idealised stretching squares model always assume finite values. This means that the structure is never 'locked' when stretched or sheared (apart for $\theta = 90^{o}$ when shearing on-axis).

This behaviour of the moduli is once again in contrast with that of the idealised 'rotating squares' model, since in the latter, the Young's modulus (i) is always isotropic (although this is dependent on θ) and (ii) assumes an infinite value when $\theta = 90^{o}$ which corresponds to the point when the structure is fully opened and hence locked. Furthermore, the rotating squares model always exhibits an infinite shear modulus, i.e. cannot be sheared.

Another very interesting property of this 'stretching squares' model is the behaviour of the Poisson's ratio for the fully closed structure ($\theta = 0^{o}$ or $\theta = 180^{o}$), when ν_{12}^{ζ} becomes isotropic with a value of zero. Such behaviour, although not a very common property, is of great practical importance since materials having a zero Poisson's ratio can be useful in many practical applications (e.g. they may be easily inserted into / removed from crevices).[5] Such an important property cannot be obtained from the idealised rotating squares model which only affords Poisson's ratios of -1.

Another difference between the properties of the idealised 'rotating squares' model [35–37] and the 'stretching squares' model presented here is that, whilst those of the idealised 'rotating squares' model were strain independent, the mechanical properties of the 'stretching squares' model presented here is strain dependent. In this respect, it is important to highlight that the expressions derived here represent the Poisson's ratios and moduli at infinitesimally small strains. In fact, one should note that when these systems are deformed, the squares change shape to become rectangles, in which case the expressions derived here for squares would not apply any more.

Before we conclude this discussion it is important to highlight the point that the expressions for the Poisson's ratio derived here are independent of the sizes of the squares.[6] This suggests that this 'stretching squares' model can be implemented at any scale, ranging from the macroscale to the micro- or nano- (molecular level) scale where the system may be treated as a material. Thus, we expect that this model is likely to be of use to scientists and engineers who may wish to design real materials which mimic the behaviour of this idealised model, or to help them understand the behaviour of existing systems which may exhibit similar properties (e.g. similar nano- or microstructures) to the one described here. In this respect, we note that as discussed elsewhere [35–37] there are various inorganic crystalline materials which are characterised by

[4] The shear modulus approaches infinity when the denominator of Eq. 37 is equal to zero, something that is attained when $1 + \cos(4\zeta)\sin(\theta) = 0$. This equation is satisfied if $\sin(\theta) = 1$ and $\cos(4\zeta) = -1$ thus giving as possible solutions $\theta = 90^{o}$ and $\zeta = \pm 45^{o}, \pm 135^{o}$.

[5] The best known example of a material which exhibits zero Poisson's ratios is cork. Such materials are important as they do not get thinner or fatter when stretched or compressed and this makes cork ideal for wine-bottle stoppers. As discussed by Prof. R.S. Lakes "The cork must be easily inserted and removed, yet it also must withstand the pressure from within the bottle. Rubber, with a Poisson's ratio of 0.5, could not be used for this purpose because it would expand when compressed into the neck of the bottle and would jam. Cork, by contrast, with a Poisson's ratio of nearly zero, is ideal in this application" [46].

[6] All elastic structures can be implemented on different size scales since it has long been known that elasticity theory has no length scale.

having the geometry modelled as a projection of a plane, e.g. systems involving octahedrally coordinated atoms [47], or zeolites.

In such real systems, one is likely to find that the idealised behaviour, where deformations occur solely through the stretching mechanism, or indeed through any other idealised mono-mode mechanism (e.g. the idealised 'rotating squares' model where the squares are perfectly rigid and simply rotate relative to each other), is too simplistic and idealistic to capture the complex deformations that occur when materials are subjected to a load. Instead, one would expect that the stretching mode of behaviour will be accompanied by other modes of deformations, which could also result in auxetic behaviour. The overall properties exhibited by such multi-mode systems can be derived using the principle of superposition and the resultant behaviour would depend on the relative extent at which each mechanism operates. Thus, for example, a system which deforms through concurrent stretching and relative rotations of the squares will tend to become more isotropic with Poisson's ratio close to -1 as the rotating squares mechanism starts to dominate as discussed elsewhere [48, 49].

Conclusion

In this work we have analysed a new way of achieving negative Poisson's ratio for a 2D system constructed from squares. In particular, we discussed a system based on the same structure as the 'rotating squares' model with the difference that instead of having the squares rotate relative to one another, the system now deforms through the elongation or shortening of the sides of the squares, i.e. through a 'stretching' mechanism. We derived analytical expression for the on-axis Poisson's ratios and Young's moduli for two orientations of this system, which are at 45° to each other, and combined these expressions to obtain the on-axis shear moduli thus deriving the full (3 × 3) compliance matrices which could then be transformed using standard off-axis techniques to obtain the off-axis properties. Whilst this way of deriving the shear properties of the system is in itself innovative (see Appendix A), the most important result is that we have been able to show that this idealised system will always show some degree of auxeticity, a highly desirable property, unless the angles between the squares is 0° in which case the system exhibits isotropic Poisson's ratios of zero.

We showed that the behaviour of this system is very different from that of the 'rotating squares'. In fact, in the case of the stretching mechanism, the Poisson's ratio has been shown to exhibit a sinusoidal variation with the direction of loading and at maximum auxeticity its value can range between 0 and -1, depending on the angle between the squares. This is much different from the idealised rotating squares system which always exhibits Poisson's ratios of -1.

We hope that the interesting results yielded by the analytical study of this novel method for generating auxetic behaviour from connected squares will be of help to scientists and engineers in their quest for the design and synthesis of new materials which can be tailor-made to meet specific mechanical demands, in particular, when producing new auxetic materials.

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Appendix A: Derivation of the relationship between the shear modulus and the Poisson's ratio and Young's moduli in two orientations at 45° to each other for systems with "2D cubic" symmetry

If the tensile σ'_i and shear τ'_{12} stresses in a coordinate system Ox'_i are known, it is possible to obtain the tensile σ_i and shear τ_{12} stresses in a coordinate system Ox_i rotated at an angle ζ with the Ox'_1-axis through the following standard transformation equations [44]:

$$\sigma_1 = \frac{\sigma'_1 + \sigma'_2}{2} + \frac{\sigma'_1 - \sigma'_2}{2}\cos(2\theta) + \tau'_{12}\sin(2\theta)$$
$$\sigma_2 = \frac{\sigma'_1 + \sigma'_2}{2} - \frac{\sigma'_1 - \sigma'_2}{2}\cos(2\theta) - \tau'_{12}\sin(2\theta)$$
$$\tau_{12} = -\frac{\sigma'_1 - \sigma'_2}{2}\sin(2\zeta) + \tau'_{12}\cos(2\zeta) \quad (38)$$

From these, it is possible to show that if the coordinate system Ox_i is rotated at an angle of $\zeta = 45°$ to the Ox'_1-axis, and $\sigma'_1 = -\sigma'_2$ while $\tau'_{12} = 0$, the tensile and shear stresses in the Ox_i system are $\sigma_1 = \sigma_2 = 0$ and $\tau_{12} = \sigma'_2$ [44]. Applying this to Orientations I and II, we can conclude that applying only the tensile strains $\sigma_1^{II} = -\sigma_2^{II}$ in Orientation II is equivalent to having a shear stress τ_{12}^{I} of magnitude σ_2^{II} in Orientation I.

Also the shear strain γ_{12} in the coordinate system Ox_1 is related to the strains ε'_i along the Ox'_i-axis and the shear strain γ'_{12} through equation [44]:

$$\gamma_{12} = -\left(\varepsilon'_1 - \varepsilon'_2\right)\sin(2\zeta) + \gamma'_{12}\cos(2\zeta) \tag{39}$$

Thus the value of the shear strain in Orientation I is related only to the strains $\varepsilon_i^{\mathrm{II}}$ in Orientation II via:

$$\gamma_{12}^{\mathrm{I}} = \varepsilon_2^{\mathrm{II}} - \varepsilon_1^{\mathrm{II}} \tag{40}$$

The terms $\varepsilon_1^{\mathrm{II}}$ and $\varepsilon_2^{\mathrm{II}}$ are themselves linked to the applied stresses in Orientation II through:

$$\begin{aligned}\varepsilon_1^{\mathrm{II}} &= s_{11}^{\mathrm{II}}\sigma_1^{\mathrm{II}} + s_{12}^{\mathrm{II}}\sigma_2^{\mathrm{II}} + s_{13}^{\mathrm{II}}\tau_{12}^{\mathrm{II}}\\ \varepsilon_2^{\mathrm{II}} &= s_{21}^{\mathrm{II}}\sigma_1^{\mathrm{II}} + s_{22}^{\mathrm{II}}\sigma_2^{\mathrm{II}} + s_{23}^{\mathrm{II}}\tau_{12}^{\mathrm{II}}\end{aligned} \tag{41}$$

So, if we now set $\sigma_1^{\mathrm{II}} = -\sigma_2^{\mathrm{II}}$ and $\tau_{12}^{\mathrm{II}} = 0$ to obtaining a pure shear $\tau_{12}^{\mathrm{I}} = \sigma_2^{\mathrm{II}}$ in Orientation I, we see that these equations reduce to:

$$\varepsilon_1^{\mathrm{II}} = -\varepsilon_2^{\mathrm{II}} = \left(s_{12}^{\mathrm{II}} - s_{11}^{\mathrm{II}}\right)\tau_{12}^{\mathrm{I}} \tag{42}$$

Substituting Eq. 42 in Eq. 40 gives:

$$\gamma_{12}^{\mathrm{I}} = 2\tau_{12}^{\mathrm{II}}\left(s_{11}^{\mathrm{II}} - s_{12}^{\mathrm{II}}\right) \tag{43}$$

The shear modulus for Orientation I is thus derived as:

$$G_{12}^{\mathrm{I}} = \frac{\tau_{12}^{\mathrm{I}}}{\gamma_{12}^{\mathrm{I}}} = \frac{1}{2\left(s_{11}^{\mathrm{II}} - s_{12}^{\mathrm{II}}\right)} \tag{44}$$

or in terms of the Poisson's ratios and Young's moduli in Orientation II:

$$G_{12}^{\mathrm{I}} = \frac{E_1^{\mathrm{II}}}{2\left(1 + v_{12}^{\mathrm{II}}\right)} \tag{45}$$

Using analogous arguments, it can be shown that:

$$G_{12}^{\mathrm{II}} = \frac{E_1^{\mathrm{I}}}{2\left(1 + v_{12}^{\mathrm{I}}\right)} \tag{46}$$

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Appendix B

Table 1 A list of the expressions of the Poisson's ratio, Young's and shear moduli at the turning points in the off-axis plots and the corresponding values of the second derivative

$f(\zeta)$	ζ/deg.	Values of $f(\zeta)$ at the turning point	Second derivative, i.e. $\frac{\partial^2 f}{\partial \zeta^2}$	Nature of turning point
$v_{ij}(\zeta)$	$\pm n(90^{\mathrm{o}})$	$v_{12}^{\mathrm{I}} = -\sin(\theta)$	$8\sin(\theta)[1+\sin(\theta)] \geq 0$	Min
	$\pm(2n+1)(45^{\mathrm{o}})$	$v_{ij}^{\mathrm{II}} = 0$	$-\frac{8\sin(\theta)}{1+\sin(\theta)} \leq 0$	Max
$E_i(\zeta)$	$\pm n(90^{\mathrm{o}})$	$E_i^{\mathrm{I}} = \frac{2K_s}{z}$	$-\frac{16\sin(\theta)K_s}{z} \leq 0$	Max
	$\pm(2n+1)(45^{\mathrm{o}})$	$E_i^{\mathrm{II}} = \frac{2K_s}{z[1+\sin(\theta)]}$	$\frac{16\sin(\theta)K_s}{[1+\sin(\theta)]^2 z} \geq 0$	Min
$G_{12}(\zeta)$	$\pm n(90^{\mathrm{o}})$	$G_{12}^{\mathrm{I}} = \frac{K_s}{z[1+\sin(\theta)]}$	$\frac{16\sin(\theta)K_s}{[1+\sin(\theta)]^2 z} \geq 0$	Min
	$\pm(2n+1)(45^{\mathrm{o}})$	$G_{12}^{\mathrm{II}} = \frac{K_s}{z[1-\sin(\theta)]}$	$-\frac{16\sin(\theta)K_s}{[1-\sin(\theta)]^2 z} \leq 0$	Max

Note that n is an integer

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